Tutorial on the Physics of Complex Systems

Exersice 18 “Replica trick”

---

d)

Expand $Z^n(\beta, J)$ to $O(\beta^4)$ for $n \in \{1, 2, 3, 4, 5\}$.

```
z[β_, j_] := 4 Cosh[β j]
zn[n_] := Series[z[β, j]^n, {β, 0, 4}]
```

In[10]:=
```
Table[zn[n], {n, 1, 5}] // TableForm
```

Out[10]//TableForm=

$$4 + 2\, j^2\, \beta^2 + \frac{j^4\, \beta^4}{6} + O[\beta]^5$$
$$16 + 16\, j^2\, \beta^2 + \frac{16\, j^4\, \beta^4}{3} + O[\beta]^5$$
$$64 + 96\, j^2\, \beta^2 + 56\, j^4\, \beta^4 + O[\beta]^5$$
$$256 + 512\, j^2\, \beta^2 + \frac{1280\, j^4\, \beta^4}{3} + O[\beta]^5$$
$$1024 + 2560\, j^2\, \beta^2 + \frac{8320\, j^4\, \beta^4}{3} + O[\beta]^5$$

In[47]:=
$$p[j\_] := \frac{1}{\sqrt{2\pi\sigma^2}} \, \mathrm{Exp}\left[-\frac{j^2}{2\sigma^2}\right]$$
$$\mathrm{averageZ}[zz\_] := \int_{-\infty}^{+\infty} p[j]\, zz\, dj$$

lhs contains $\langle Z^n(\beta, J)\rangle_J$ for $n \in \{1, 2, 3, 4, 5\}$ approximated $O(\beta^4)$ like above.

In[43]:=
```
lhs = Table[Refine[averageZ[zn[n]], {Element[σ, Reals], σ > 0}], {n, 1, 5}];
TableForm[lhs]
```

Out[44]//TableForm=

$$\frac{1}{2}\left(8 + 4\,\beta^2\,\sigma^2 + \beta^4\,\sigma^4\right)$$
$$16\left(1 + \beta^2\,\sigma^2 + \beta^4\,\sigma^4\right)$$
$$8\left(8 + 12\,\beta^2\,\sigma^2 + 21\,\beta^4\,\sigma^4\right)$$
$$256\left(1 + 2\,\beta^2\,\sigma^2 + 5\,\beta^4\,\sigma^4\right)$$
$$128\left(8 + 20\,\beta^2\,\sigma^2 + 65\,\beta^4\,\sigma^4\right)$$

rhs contains $2^{2n} + 2^{2n-1}\, n\, \sigma^2\, \beta^2 + 2^{2n-3}\,(3n-2)\,\sigma^4\,\beta^4$ for $n \in \{1, 2, 3, 4, 5\}$.

```
rhs = Table[2^(2 n) + 2^(2 n-1) n σ^2 β^2 + 2^(2 n-3) n (3 n - 2) σ^4 β^4, {n, 1, 5}] ;
TableForm[rhs]
```

Out[42]//TableForm=

$4 + 2\,\beta^2\,\sigma^2 + \frac{\beta^4\,\sigma^4}{2}$

$16 + 16\,\beta^2\,\sigma^2 + 16\,\beta^4\,\sigma^4$

$64 + 96\,\beta^2\,\sigma^2 + 168\,\beta^4\,\sigma^4$

$256 + 512\,\beta^2\,\sigma^2 + 1280\,\beta^4\,\sigma^4$

$1024 + 2560\,\beta^2\,\sigma^2 + 8320\,\beta^4\,\sigma^4$

In[46]:=
```
Table[FullSimplify[lhs[[n]]] == FullSimplify[rhs[[n]]], {n, 1, 5}] // TableForm
```

Out[46]//TableForm=
```
True
True
True
True
True
```

The expressions are identical, therefore $\langle Z^n(\beta, J)\rangle_J = 2^{2n} + 2^{2n-1}\, n\, \sigma^2\, \beta^2 + 2^{2n-3}\,(3\,n - 2)\,\sigma^4\,\beta^4$ up to order $O(\beta^4)$.